VECTOR-VALUED ANALYTIC FUNCTIONS(1)

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1. **Introduction.** The functions to be studied here take values in a complex metric linear space. The metric topology is not assumed to be locally convex. Consequently, important classical theorems about the behavior of functions are lost. Two specific instances are as follows. First, a nonconstant function may have a derivative which is everywhere equal to 0, the zero or neutral element of the linear space. This seems to have been first remarked upon by Paul Levy [4, p. 57]. Second, a continuous function with domain the closed unit interval may fail to be integrable with respect to Lebesgue measure.

The metric of the linear space will not be used explicitly until §5 in connection with the metric maximum modulus property. However, a countable base for neighborhoods of 0 is used where convenient.

The central result is given in Theorem 4.2.5. It characterizes the class of functions defined here as consisting of those which can be factored locally into the composition of an analytic function with values in a Banach space and a continuous linear transformation from the Banach space into the original metric linear space. Much is therefore known of the local behavior of such functions.

The basic definitions and notational conventions to be used are given in §2 below.

2. **Preliminaries.** Every linear space mentioned here has the complex field C as groundfield. Topologies for linear spaces, other than those which may be explicitly defined or constructed, are assumed to be metrizable, but need not be locally convex. The linear spaces are complete with respect to their original topologies. The class of open balanced neighborhoods of the neutral or zero element 0 is denoted by $N(0)$. The closure of a set A is denoted by cl A or cl(A) as seems appropriate. If it is desirable to specify the topology T associated with the closure operation, the notation T-cl A or T-cl(A) is used. The set of all positive integers is denoted by ω; a sequence is denoted by $\{x_n: n \in \omega\}$ or $\{x_n: n = 1, 2, \cdots\}$ or other obvious variations. This minor abuse of standard notation should not lead to confusion.

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The convex hull of a set A in a linear space is denoted by $\operatorname{cvx} A$ or $\operatorname{cvx}(A)$ as seems appropriate. The linear span of A is denoted by LA or $L(A)$, and the balanced convex hull of A is denoted by $\operatorname{bx} A$ or $\operatorname{bx}(A)$. Additional notation is introduced later as needed.

The definitions fundamental to the present study are as follows:

2.1. DEFINITION. Let X be a set and let E be a linear space. A function $f\colon X \to E$ is said to be of, or to have, finite rank if and only if $L(f[X])$ is finite dimensional. Otherwise f is said to have infinite rank.

2.2. DEFINITION. Let X be a set and let E be a topological linear space. A sequence of functions $f_n\colon X \to E$ is said to be ultra convergent to a function $f_0\colon X \to E$ if and only if for every $W \in N(0)$ there exists an n' such that if $n \geqq n'$ then $\operatorname{cvx}((f_n - f_0)[X]) \subset W$. The sequence $\{f_n\colon n = 1, 2, \cdots\}$ is said to be ultra Cauchy if and only if for every $W \in N(0)$ there exists an n' such that if $m \geqq n'$ and $n \geqq n'$ then $\operatorname{cvx}((f_n - f_m)[X]) \subset W$.

2.3. DEFINITION. Let $G \subset C$ be open and let E be a topological linear space. A function $f\colon G \to E$ is said to be differentiable at $z \in G$ if and only if the limit $\lim_{w=0}(1/w)(f(z+w) - f(z))$ exists. A function is said to be differentiable on G if and only if it is differentiable at every point of G.

2.4. DEFINITION. Let $G \subset C$ and let E be a topological linear space. A function $f\colon G \to E$ is said to be class A_0 on G if and only if f has finite rank and f is differentiable. The set of all such functions is denoted by $A_0(G, E)$. The qualifiers G or E may be omitted at times.

2.5. DEFINITION. Let $G \subset C$ be open and let E be a topological linear space. A function $f\colon G \to E$ is said to be of class A on G if and only if there exists a sequence of functions $f_n \in A_0(G, E)$ which is ultra convergent to f on every compact subset of G. The set of all such functions is denoted by $A(G, E)$.

The class of "analytic" functions studied here, of course, is the class $A(G, E)$. It is natural to ask if the strengthened convergence of the definition is simply convergence with respect to a stronger, locally convex topology. The answer is believed to be "no".

There need not exist a single topology for all of E such that ultra convergence is equivalent to uniform convergence in that stronger topology.

3. **Integration of vector-valued functions.** Much of the basic theory of functions of class A is based upon the Cauchy integral formula. Hence an integration theory must be available. A generalization of the Birkhoff integral of strongly measurable functions could be carried out; however, such an integral is more general than is required here. For simplicity, a specialized integral adequate for present purposes is used instead.

Throughout §3, $(X, \boldsymbol{B}, \mu)$ will denote a fixed measure space; for convenience, $\mu(X) = 1$ is assumed. The symbol E will denote a fixed complete metric linear space.

3.1. DEFINITION. A function $f: X \to E$ is said to be simple if and only if there exists a finite set of vectors $e_j \in E, j = 1, \cdots, n$, and sets $S_j \in \boldsymbol{B}, j = 1, \cdots, n$, such that $\bigcup\{S_j: j = 1, \cdots, n\} = X$ and $f(x) = e_j$ if $x \in S_j$.

Note that a simple function is measurable and has finite rank.

3.2. DEFINITION. Let $f: X \to E$ be simple, with the representation

$$f(x) = \sum\{\chi(x; S_j)e_j: j = 1, \cdots, n\}.$$

The integral $\int f\, d\mu$ of f is defined to be

$$\sum\{\mu(S_j)e_j: j = 1, \cdots, n\}.$$

3.3. DEFINITION. A function $f: X \to E$ is said to be integrable if and only if there exists a sequence of simple functions $f_n: X \to E$ which is ultra convergent to f on X.

The pattern of the theory is probably clear from Definition 3.3; the example at the end of the section should be noted, however. For the sake of completeness, the integration theory will be developed to the extent necessary for the present work. The propositions needed and their proofs are all rather obvious, so a minimum of detail and no supplemental explanation are given.

3.4. LEMMA. *Let $f_n: X \to E$ be simple and let the sequence $\{f_n: n \in \omega\}$ be ultra convergent to $f: X \to E$. Then the sequence of integrals $\{\int f_n d\mu: n \in \omega\}$ is Cauchy.*

Proof. The lemma follows from the observations that

$$\int f_n d\mu - \int f_m d\mu \in \mathrm{cvx}((f_n - f_m)[X])$$

and

$$\mathrm{cvx}((f_n - f_m)[X]) \subset \mathrm{cvx}((f_n - f)[X]) - \mathrm{cvx}((f_m - f)[X])$$

for all n and m.

3.5. LEMMA. *Let the sequences of functions $f_n: X \to E$ and $g_n: X \to E$ be ultra convergent to $f: X \to E$ and $g: X \to E$ respectively, and let the sequence of scalars λ_n converge to λ. Then the sequence $\{f_n + \lambda_n g_n: n \in \omega\}$ is ultra convergent to $f + \lambda g$.*

Proof. This follows from the fact that both a scalar multiple of a convex set and the sum of two convex sets are themselves convex.

3.6. PROPOSITION. *Let the function $f: X \to E$ be integrable, and let the sequences of simple functions $f_n: X \to E$ and $g_n: X \to E$ each be ultra convergent to f. Then $\lim_{n=0} \int f_n d\mu = \lim_{n=0} \int g_n d\mu$.*

Proof. The sequence $\{f_n - g_n: n \in \omega\}$ is ultra convergent to 0.

The following is therefore a legitimate definition.

3.7. Definition. Let $f : X \to E$ be integrable and let $\{f_n : n \in \omega\}$ be any sequence of simple functions which is ultra convergent to f. Then the integral of f is defined by

$$\int f\,d\mu = \lim_{n=\infty} \int f_n\,d\mu.$$

If $B \in \boldsymbol{B}$ then $\int_B f\,d\mu$ is defined by

$$\int_B f\,d\mu = \int \chi(x;B)f(x)\,d\mu(x).$$

3.8. Proposition. *The integral in Definition 3.5 is a linear function of the integrand and a finitely additive E-valued set function for a fixed integrand.*

Proof. This follows from the obvious validity of these facts for simple functions.

3.9. Proposition. *Let the sequence of integrable functions $f_n : X \to E$ be ultra convergent to the function $f : X \to E$. Then f is integrable, and*

$$\int f\,d\mu = \lim_{n=\infty} \int f_n\,d\mu.$$

Proof. For each n, let the sequence of simple functions $g_{nk} : X \to E$ be ultra convergent to $f_n : X \to E$. It suffices to show that there exists a sequence of integers $k(n)$ such that $\{g_{nk(n)} : n \in \omega\}$ is ultra convergent to f. But

$$\begin{aligned}(g_{nk} - f)[X] &\subset (g_{nk} - f_n)[X] - (f_n - f)[X]\\ &\subset \operatorname{cvx}((g_{nk} - f_n)[X]) - \operatorname{cvx}((f_n - f)[X]).\end{aligned}$$

Let $\{W_j : j \in \omega\} \subset N(0)$ be a basis for neighborhoods at 0. It suffices to select $k(n)$ so that there holds $\operatorname{cvx}((g_{nk(n)} - f_n)[X]) \subset W_n$ for all n. For, if $U \in N(0)$ let $V \in N(0)$ be such that $V + V \subset U$. Let n_1 be so large that if $n \geqq n_1$ then $W_n \subset V$ and let n_2 be so large that if $n \geqq n_2$ then $\operatorname{cvx}((f_n - f)[X]) \subset V$. If $n \geqq \max(n_1, n_2)$, then $\operatorname{cvx}((g_{nk(n)} - f)[X]) \subset \operatorname{cvx}((g_{nk(n)} - f_n)[X]) - \operatorname{cvx}((f_n - f)[X]) \subset V + V \subset U$.

From the preceding results, it is clear that the measure μ may be replaced by any complex-valued, countably additive set function (called here a complex measure) of finite total variation. This is henceforth considered accomplished. The following obvious fact is recorded here without proof for future reference:

3.10. Proposition. *Let $f : X \to E$ be integrable and let α be a complex measure. Let α have the representation $\alpha = \alpha_1 + i\alpha_2$, with α_1 and α_2 real. Let $\|\alpha\| = \|\alpha_1\| + \|\alpha_2\|$, where $\|\alpha_j\|$ denotes the total variation of α_j, $j = 1,2$. Then*

$$\int f\,d\alpha \in \|\alpha\| \operatorname{cl} \operatorname{bx}(f[X]).$$

It seems appropriate to end this section with an example indicative of the kind of limitation needed to obtain a theory of integration in the present context. In particular, the example shows that a continuous mapping of the unit interval into a complete metric linear space may not be integrable with respect to Lebesgue measure for any reasonable definition of integration.

3.1.1. EXAMPLE. Let M denote the space of all measurable, complex-valued functions defined on the unit interval $[0,1]$, with functions equal almost everywhere identified. The topology of M is that of convergence in measure. Let $f:[0,1]\to M$ be defined by $[f(a)](x) = 1/|x-a|$. Then f is not integrable with respect to Lebesgue measure.

DEMONSTRATION. Consider the subdivision of $[0, 1]$ generated by the points k/n, $k = 0,1,\cdots,n$. Let a_{kn} be any point such that $(k-1)/n < a_{kn} < k/n$, $k = 1,\cdots,n$. Let U_{jn} denote the interval

$$\{x:(j-1)/n \leqq x \leqq j/n\}, \qquad j = 1,\cdots,n.$$

For $x \in U_{jn}$, there holds:

$$[f(a_{kn})](x) \geqq |((j-1)/n) - (k/n)|^{-1} = |n/(j-k-1)|,$$

if $j \leqq k$, and

$$[f(a_{kn})](x) \geqq |(j/n) - ((k-1)/n|^{-1} = |n/(j-k+1)|,$$

if $j > k$. But then for $x \in U_{jn}$,

$$\sum\{(1/n)\,[f(a_{kn})]\,(x):1 \leqq k \leqq n\}$$

is at least

$$\sum\left\{\frac{1}{k+1-j}:j \leqq k \leqq n\right\} + \sum\left\{\frac{1}{j+1-k}:1 \leqq k < j\right\}$$
$$\geqq 1 + 1/2 + \cdots + 1/n, \qquad \text{for any } j.$$

This shows that the function f is not integrable in the above sense. For, if it were, then $\mathrm{cvx}\, f[X]$ would be bounded; the compution shows that it is not.

It is clear that the anomalous behavior just observed will prevent the function f from being integrable with respect to any "reasonable" integration theory, in spite of the fact that f is continuous, as an M-valued function.

4. **Functions of class** A. The fundamental properties of functions of class A are developed here. The first few results establish some properties of functions of class A_0 to be used later. Throughout this section, the symbol E denotes a complete metric linear space.

4.1. *Functions of class* A_0. It is obvious that the Cauchy integral formulas apply to functions of class A_0. The integrals of the form $\int_\gamma f(\zeta)d(\zeta)$, where γ denotes a rectifiable curve, are understood as usual to mean $\int_0^1 f(\gamma(t))d\gamma(t)$ for a suitable parameterization of γ. Integration can be understood to be with respect to the complex measure associated with the function of bounded variation $\gamma(t)$.

4.1.1. Lemma. *Let $W \subset C$ be open and let $f: W \to E$ be of class A_0. Let G be a Cauchy domain with* cl $G \subset W$. *Then for any $z \in G$ and integer $k \geqq 0$ there holds*

$$f^{(k)}(z) = (k!/2\pi i) \int_{\partial G} (\zeta - z)^{-k-1} f(\zeta) d\zeta.$$

Proof. Under the usual meaning of the integral, no verification is needed; the range is included in a finite-dimensional subspace. But, since the topology of that subspace is locally convex, so that uniform and ultra convergence coincide, the formulas hold for the integral of Definition 3.7 as well.

It is interesting and suggestive to observe that the functions of class A_0 have a linear-algebraic characterization independent of the topology of E.

4.1.2. Theorem. *Let $G \subset C$ be a connected open set. A function $g: G \to E$ belongs to the class A_0 if and only if for every linear form ψ in the algebraic dual E^f of E the function $\psi \circ g: G \to C$ is analytic.*

Proof. It is clear that if $g \in A_0(G, E)$ then $\psi \circ g$ is analytic. It is also clear that if the linear span $L(g[G])$ is finite dimensional and if $\psi \circ g$ is analytic for every $\psi \in E^f$, then g is differentiable. It suffices, then, to show that if $\psi \circ g$ is analytic for all $\psi \in E^f$ then $L(g[G])$ is finite dimensional.

Note first that g is $w(E, E^f)$ continuous. It follows that for any $z \in G$ and any closed disk $B_r(z)$ of radius r and center z, with $B_r(z) \subset G$, there holds that $g[B_r(z)]$ is compact and so spans a finite-dimensional subspace $L(g[B_r(z)])$. But an obvious analytic continuation argument then yields that $g[G] \subset L(g[B_r(z)])$, and the proof is complete.

4.2. *Functions of class A.* The central result of this section is the characterization of functions of class A given in Theorem 4.2.5. The basic tool is a development of a technique used by Grothendieck [2]. The preliminary results formalize the tool; the central result is then applied to the study of some local properties of functions of class A.

4.2.1. Lemma. *Let $g \in A(G, E)$ and let $K \subset G$ be compact. Then* cl cvx $g[K]$ *and* cl bx $g[K]$ *are bounded.*

Proof. Let $g_n \in A_0(G, E)$ for all $n \in \omega$ and let $\{g_n: n \in \omega\}$ be ultra convergent to g. It is clear that cl cvx $g_n[K]$ is compact and therefore bounded for all n.

Let $W \in N(0)$; let $V \in N(0)$ be such that $V + V \subset W$. For n sufficiently large, there holds $\mathrm{cvx}\,(g_n - g)[K] \subset V$. Therefore, since $g[K] \subset \mathrm{cvx}(g - g_n)[K] + \mathrm{cvx}\, g_n[K]$ and there exists a $\rho > 0$ such that $\mathrm{cvx}\, g_n[K] \subset \rho V$, it follows that $\mathrm{cvx}\, g[K] \subset V + \rho V$. Let $\varepsilon = \max(\rho, 1)$. Then $\mathrm{cvx}\, g[K] \subset \varepsilon V + \varepsilon V \subset \varepsilon W$, and the proof is complete.

4.2.2. Lemma. *Let $g \in A(G, E)$ and let $K \subset G$ be compact. Let L denote the linear span of* cl bx $(g[K])$ *and let p denote the Minkowski functional of* cl bx $(g[K])$. *Then (L, p) is a Banach space.*

Proof. The functional p is a norm (positive definite) by virtue of the boundedness of cl bx $(g[K])$. Completeness with respect to the original metric then implies completeness of L with respect to p (see, e.g., [1, p. 11, Proposition 8]).

4.2.3. Lemma. *Let $G \subset C$ be open, and let U be a Cauchy domain such that* cl $U \subset G$. *Let $g \in A(G, E)$. Then $\int_{\partial U} [1/(w-z)^k] g(w)dw$ exists for all $z \notin \partial U$ and any integer $k > 0$. Moreover, if $z \in U$, then*

$$(1/2\pi i) \int_{\partial U} [1/(w-z)] g(w)dw = g(z).$$

Proof. The integrands are integrable, by virtue of Proposition 3.7, using an approximating sequence of functions of class A_0. That same proposition yields the Cauchy integral formula since the latter is known to hold for functions of class A_0.

4.2.4. Theorem. *Let $G \subset C$ be open and let U be a Cauchy domain with* cl $U \subset G$. *Let $g \in A(G, E)$, and define $f: U \to E$ by*

$$f(z) = (1/2\pi i) \int_{\partial U} [1/(w-z)^2] g(w)dw.$$

Let L be the linear span of cl bx $(g[\partial U])$, *and let p be the Minkowski functional of* cl bx$(g[\partial U])$ *on L. Then for any $z \in U$ there holds:*

$$\lim_{w=0} p([(g(z+w) - g(z))/w] - f(z)) = 0.$$

Proof. First, Proposition 3.10 shows that $f(z) \in L$ if $z \in U$. Lemma 4.2.3 yields

$$[g(z+w) - g(z)]/w = (1/2\pi i) \int_{\partial U} 1/[(\zeta - z)(\zeta - z - w)] g(\zeta)d\zeta$$

and

$$[g(z+w) - g(z)]/w - f(z) = (1/2\pi i) w \int_{\partial U} 1/[(\zeta - z)^2(\zeta - z - w)] g(\zeta)d\zeta.$$

Basic classical manipulations show that

$$p([g(z+w) - g(z)]/w - f(z)) = O(w),$$

and the proof is complete.

It is observed here that the validity of the Cauchy integral formula follows immediately.

The characterization theorem immediately below is suggested by and follows readily from Theorem 4.2.4.

4.2.5. Theorem. *Let $G \subset C$ be open and connected. A function $g: G \to E$*

belongs to $A(G, E)$ *if and only if for every* $z \in G$ *there exist an open disk* $D(z) \subset G$, *a Banach space* $L(z)$, *a continuous linear transformation* $T_z: L(z) \to E$, *and an analytic function* $f_z: D(z) \to L(z)$ *such that* $g \mid D(z) = T_z \circ f_z$.

Proof. Necessity is immediate from 4.2.4. To establish sufficiency, a sequence of functions $g_n \in A_0(G,E)$ must be exhibited which is ultra convergent to g on every compact subset of G.

There exists an increasing sequence of Cauchy domains U_n such that $\mathrm{cl}\, U_n$ is compact, $\mathrm{cl}\, U_n \subset U_{n+1} \subset G$, and $\bigcup\{U_n: n \in \omega\} = G$. Then, for each n, there exists a finite set $Z_n \subset \mathrm{cl}\, U_n$ such that

$$\mathrm{cl}\, U_n \subset \bigcup\{D(z): z \in Z_n\}.$$

Now, there exist open disks $D'(z)$, with center z, such that $\mathrm{cl}\, D'(z) \subset D(z)$, and $\mathrm{cl}\, U_n \subset \bigcup\{D'(z): z \in Z_n\}$. Clearly $\mathrm{cl\,bx}\, g[\mathrm{cl}\, D'(z)]$ is bounded in E for each $z \in Z_n$, since $S_z = \mathrm{cl\,bx}\, f_z[\mathrm{cl}\, D'(z)]$ is compact in $L(z)$. Let

$$S_n = \mathrm{cl\,bx}\, g[\bigcup\{\mathrm{cl}\, D'(z): z \in Z_n\}]$$

and let L_n denote $L(S_n)$, and p_n denote the Minkowski functional of S_n on L_n; (L_n, p_n) is a Banach space. Let $L'(z)$ denote $L(S_z) \subset L(z)$, and p'_z the norm associated with S_z. Then $(L'(z), p'_z)$ is a Banach space, for each $z \in Z_n$, and $T_z \mid L'(z)$ is continuous, with the range considered as (L_n, p_n). Let $V_n = \bigcup\{D'(z): z \in Z_n\}$. Then clearly $g \mid V_n: V_n \to (L_n, p_n)$ is analytic. It follows that a sequence of polynomials $f_{mn}: C \to L_n$ exists such that $p_n(f_{mn} - g) \to 0$ uniformly on $\mathrm{cl}\, U_n$. The metrizability of E permits the selection of a sequence of integers $m(n)$ such that $\{f_{m(n)n}: n \in \omega\}$ is ultra convergent to g on every $\mathrm{cl}\, U_n$; this completes the proof.

4.2.5.1. Corollary. *A function* $f: G \to E$ *locally of class* A *on* G *belongs to* $A(G, E)$.

4.2.5.2. Corollary. *Let* $G \subset C$ *be open, let* $g \in A(G,E)$, *and let* $z \in G$. *Then there exists an open disk* $D(z) \subset G$ *such that* g *has a Taylor series expansion* $\sum(w - z)^k a_k$ *which is ultra convergent to* g *on compact subsets of* $D(z)$. *If* $D'(z)$ *is any open disk such that* $\mathrm{cl}\, D'(z) \subset D(z)$, *then the coefficients* a_k *belong to the linear span of* $\mathrm{cl\,bx}\, g[D'(z)]$.

It is obvious from Theorem 4.2.5 that a function of class A has a local behavior which is the same as that of an analytic function with values in a Banach space, at least as far as linear topological properties are concerned.

4.2.6. Theorem. *Let* D *denote the open unit disk with center* 0 *in* C, *and let* D_r *and* B_r *respectively denote the open and closed disks with center* 0 *and radius* r *in* C. *Let* $f \in A(D, E)$ *be of infinite rank, let* $r \in (0, 1)$, *and let* $f_r = f \mid B_r$. *Let* $S_r = \mathrm{cl\,bx}\, f[B_r]$, *and let* $L_r = L(S_r)$. *Let* p_r *denote the Minkowski functional* *of S_r on L_r. Then $f_r\colon B_r \to (L_r, p_r)$ is analytic on D_r, has no isolated singularities on ∂B_r, and is not continuous on ∂B_r.*

Proof. The fact that f_r is analytic on D_r follows immediately from Theorem 4.2.5. Observe that if f_r were continuous on ∂B_r, then $f[\partial B_r]$ would be compact in (L_r, p_r), so that S_r would also be compact in (L_r, p_r). Then L_r would be finite dimensional; but this is impossible, since f has infinite rank. It follows that f_r is not p_r-continuous on ∂B_r.

The fact that a singularity at $z_0 \in \partial B_r$ cannot be isolated is established as follows. If z_0 were isolated, a Cauchy domain W could be found such that $z_0 \in W$ and f_r is analytic as a function with values in (L_r, p_r) on $W \setminus \{z_0\}$. Then $f[\partial W] \subset L_r$, and so $f[\partial W]$ is p_r-compact. Therefore $\operatorname{cl}\operatorname{bx} f[\partial W]$ is p_r-compact. Let T_E denote the original topology of E. Since the p_r-topology is stronger than T_E on $\operatorname{cl}\operatorname{bx} f[\partial W]$, the two topologies coincide there. But f is of class A in a neighborhood of z_0, considered as an (E, T_E)-valued function. Therefore, from the Cauchy integral formulas, f_r must be analytic in a neighborhood of z_0 as an (L_r, p_r)-valued function. This completes the proof.

It is obvious that if F denotes the closure of the set of points in ∂B_r which map onto extreme points of $\operatorname{cl}\operatorname{cvx} f[\partial B_r]$, then F is compact but $f_r[F]$ is not p_r-compact; otherwise, $\operatorname{cl}\operatorname{cvx} f[\partial B_r] = \operatorname{cl}\operatorname{cvx} f[F]$ would again be p_r-compact. It is not known if every singularity of f_r on ∂B_r belongs to F. The type of singularity indicated by these considerations appears not to have been examined in any detail as yet, although such behaviour has been remarked upon by Taylor [6, p. 657].

The final topic to be examined in this section is a linear topological version of the maximum modulus principle. The symbol D is used to denote the set $\{z : z \in C.\ \&.\ |z| < 1\}$, the symbol D_r for $\{z : z \in C.\ \&.\ |z| < r\}$, and $B_r = \operatorname{cl} D_r$.

4.2.7. Lemma. *Let $0 < r < 1$ and let $g \in A(D)$. Then $\operatorname{cl}\operatorname{cvx} g[B_r]$ is the closed convex hull of its set of extreme points. Moreover, $\operatorname{cl}\operatorname{cvx} g[B_r] = \operatorname{cl}\operatorname{cvx} g[\partial B_r]$, so that every extreme point is the image of a boundary point.*

Proof. First note that if $r < t < 1$, then $g\colon D_t \to (L_t, p_t)$ is analytic (L_t and p_t have the same meaning as in Lemma 4.2.6). It follows that $g[B_r]$ is a compact subset of (L_t, p_t). Therefore, the p_t-closed convex hull of $g[B_r]$ is p_t-compact; the first assertion follows. The second assertion follows immediately from an application of the Poisson integral formula.

4.2.8. Lemma. *Let $g \in A(D)$ and let $g(0) = 0$. Let r and t be such that $0 < r < t < 1$, and let a be such that $0 \leqq a \leqq (r + t)/2r$. Then for any θ there holds $ag(re^{i\theta}) \in \operatorname{cl}\operatorname{cvx} g[B_t]$.*

Proof. Let $p(r,\theta;t,\psi)$ denote the Poisson kernel $(t^2 - r^2)/[t^2 + r^2 - 2rt\cos(\theta - \psi)]$. Let $0 \leqq b \leqq (t - r)/(t + r)$ and let $h_b = (1/[2\pi(1 - b)])[p(r,\theta;t,\psi) - b]$. Then h_b is a probability density on ∂B_r, and

$$g(re^{i\theta}) = (1-b)\int_0^{2\pi} h_b(\theta,\psi)g(te^{i\theta})d\psi,$$

since $g(0) = 0$. Therefore,

$$g(re^{i\theta})/(1-b) \in \mathrm{cl\,cvx}\, g[B_t].$$

The inequality $0 \leqq b \leqq (t-r)/(t+r)$ is equivalent to $1 \geqq (1-b) \geqq 2r/(t+r)$. The lemma follows by taking $a = 1/(1-b)$.

4.2.9. THEOREM. *Let $g \in A(D)$. Then the mapping $r \to \mathrm{cl\,cvx}\, g[Br]$ is nondecreasing, using the inclusion ordering of sets. Moreover, if there exist real numbers r and t such that $0 < r < t < 1$ and*

$$\mathrm{cl\,cvx}\, g[B_t] = \mathrm{cl\,cvx}\, g[B_r],$$

then the function g is constant.

Proof. Only the second conclusion requires proof. It may be assumed that $g(0) = 0$. Then, from Lemma 4.2.8, there exists an $a > 1$ such that a $\mathrm{cl\,cvx}\, g[B_r] \subset \mathrm{cl\,cvx}\, g[B_r]$; whence $\mathrm{cl\,cvx}\, g[B_r] = \{0\}$.

4.2.9.1. COROLLARY. *If E is a Banach space, if $g \in A(E, D)$, and if $M(r) = \sup\{\| g(z) \| : | z | \leqq r\}$, then M is a nondecreasing function of r. If the norm is strictly convex, and if g is nonconstant, then M is a strictly increasing function of r.*

It seems appropriate to end this section with a simple example of a differentiable function which is not of class A. Let M denote the class of Borel measurable complex-valued functions with domain $D = \{z : z \in C. \&. |z| < 1\}$, with functions equal almost everywhere identified. Using plane Lebesgue measure on D, give M the topology of convergence in measure. Let L denote the space of complex-valued functions with domain D which are integrable Lebesgue, and let L have the usual L_1-norm. Let $\varepsilon > 0$ be a small number, and let $D_\varepsilon(z)$ denote $\{w : w \in C \cdot \& \cdot |w - z| < \varepsilon\}$. Let $f(z)$ denote the characteristic function of $D \cap D_\varepsilon(z)$. Then $f : D \to L \subset M$ is continuous, as an L-valued function; it is easily seen that $f : D \to M$ (convergence in measure) is differentiable with respect to the complex variable z, and in fact $df/dz = 0$ identically. It is also easily seen that, although $(1/2\pi i)\int_{\partial K}[1/(\zeta - z)]f(\zeta)d\zeta$ exists for any Cauchy domain K with $\mathrm{cl}\, K \subset D$, in general it is not true that the value of the integral is $f(z)$. For example, let $z = 0$, let $\varepsilon = 1/4$, and let $K = \{\zeta : |\zeta| < 2/3\}$. Then the integral is the limit of a sequence of functions which vanish inside $\{\zeta : |\zeta| < 5/12\}$. But $f(0)$ is a function equal to 1 on $\{\zeta : |\zeta| < 1/4\}$; the integral, whatever its value might turn out to be, is not $f(0)$. (The value of the integral is essentially th function whose value at $w \in D$ is the fraction of the circumference of $\{\zeta : |\zeta| = 2/3\}$ lying inside a disk of radius 1/4 around w.)

5. **The maximum modulus principle.** Analogs of the classical maximum modulus and three circles theorems may not exist, even for functions of class A_0, for completely general invariant metrics. Two examples may be mentioned. First, consider $E = C \times C$, with the metric defined by

$$\rho((z_1, z_2), (0, 0)) = |\operatorname{Re}(z_1)|^{1/2} + |\operatorname{Im}(z_1)|^{1/2} + |\operatorname{Re}(z_2)|^{1/2} + |\operatorname{Im}(z_2)|^{1/2}.$$

Define $f : C \to E$ by $f(z) = (1, 1) + z(-1, i)$. Consider the modulus $\rho(f(z), 0)$ on the square $0 \leqq \operatorname{Re}(z) \leqq 1, 0 \leqq \operatorname{Im}(z) \leqq 1$. Elementary computations show that the maximum of the modulus occurs at the point $z = 1/2 + i/2$. For a second example, let M denote the space of measurable functions with domain the unit circle. The topology of M is that of convergence in measure. A metric yielding the topology is given by

$$\rho(f, 0) = \int_0^{2\pi} |f(\theta)|/(1 + |f(\theta)|)d\theta.$$

Define $F:\{z: |z| < 1\} \to M$ by $[F(z)](\theta) = 2 + ze^{i\theta}$. It is easy to verify that the maximum modulus principle fails for F.

However, the maximum modulus property and the related convexity property are valid in a fairly wide class of metric linear spaces, identified in Definition 5.1 below. The symbol ψ will be used consistently to denote the "norm" $\psi(x) = \rho(x,0)$.

5.1. Definition. The norm ψ on a metric linear space is said to be plurisubharmonic if and only if for all x, $y \in E$ the mapping

$$\lambda \to \psi(x + \lambda y)$$

is subharmonic on C.

The definition is due to Lelong [3], who first defined and studied plurisubharmonic functionals on finite-dimensional vector spaces. Free use will be made of results established in Lelong's fundamental paper. The properties of subharmonic functions needed here are established, for all practical purposes, in Rado's exposition [5]. The characteristic property that the mean value on the circumference of a disk is at least as large as the value at the center of the disk should be recalled; thus, for instance, any convex function defined on a region in the complex plane is subharmonic in that region. A convex norm is therefore plurisubharmonic.

The mean-value property, of course, implies a maximum property; if a nonconstant function is subharmonic in a neighborhood of a closed disk, then its maximum over the disk can only be attained on the boundary. For applications, it should be observed that a positive linear combination of subharmonic functions is subharmonic, and that if $f(z)$ is an analytic function, then for any $p>0$ the function $|f(z)|^p$ is subharmonic. Finally, it is recalled that a pointwise limit of (say) a sequence of uniformly bounded subharmonic functions is subharmonic. It follows that for any p with $0 < p \leqq 1$ the norm

$$\psi_p(f) = \int_X |f(x)|^p d\mu(x)$$

of an L_p-function on a finite measure space $(X, \boldsymbol{B}, \mu)$ is plurisubharmonic. Therefore a Hardy H_p-space also carries a plurisubharmonic norm. The most important for present application is the following:

5.2. LEMMA. *Let $\phi: C^n \to [-\infty, \infty)$ be upper semicontinuous and satisfy the condition that for fixed a, $b \in C^n$ the mapping $\lambda \to \phi(a + \lambda b)$ is subharmonic on C. Let $G \subset C$ be open and let $f: G \to C^n$ be analytic. Then*

$$\phi \circ f: G \to [-\infty, \infty)$$

is subharmonic.

The proof is given in [3] and so is omitted here. The following is immediate from Lemma 5.2:

5.3. LEMMA. *Suppose that the norm ψ is plurisubharmonic on E and that $f \in A_0(G, E)$. Then the mapping $\psi \circ f: G \to [0, \infty)$ is subharmonic.*

5.4. THEOREM. *Let the norm ψ on E be plurisubharmonic and let $f \in A(G, E)$. Then the mapping $\psi \circ f: G \to [0, \infty)$ is subharmonic.*

Proof. Let $f_n \in A_0(G, E)$ be such that the sequence $\{f_n: n \in \omega\}$ is ultra convergent to f on compact subsets of G. It is clear that $\psi \circ f_n$ converges to $\psi \circ f$ uniformly on compact subsets of G; the asserted property follows.

5.4.1. COROLLARY. *Let the norm ψ on E be plurisubharmonic, and let $f \in A(G, E)$. Let the disk $\{z: |z - z_0| \leqq R\} \subset G$, and for $0 \leqq r \leqq R$ let*

$$M(r) = \sup\{\psi(f(z)): |z - z_0| = r\}.$$

Then $M(r)$ is an increasing function of r, and, for $-\infty < x \leqq \log R$, $M(e^x)$ is a convex function of x.

Recall that the Hardy H_p-spaces carry a plurisubharmonic norm. Consequently, the proof indicated by Taylor [7] for the Hardy theorem asserting that

$$M_p(f; r) = \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

is a nondecreasing function of r and that $\log M_p(f; e^x)$ is a convex function of x whenever f is analytic in a disk $\{z: |z| \leqq R\}$ has an analog even for the case $0 < p < 1$; in fact, the attempt to extend Taylor's argument was the starting point of the present investigation.

6. **Application. The spectrum of an operator on L_p, $0 < p < 1$.** The theory developed above can be applied to show that a continuous endomorphism of an L_p-space or an H_p-space ($0 < p < 1$) has a nonempty spectrum. Since the same proof applies in either case, it is convenient to abstract the notion of a (p)-space.

6.1. DEFINITION. A metric linear space E is said to be a (p)-space ($0 < p \leqq 1$) if and only if the space is complete, the norm ψ is plurisubharmonic, and ψ satisfies the homogeneity condition,

$$\psi(\lambda x) = |\lambda|^p \psi(x), \quad \text{all } x \in E, \quad \text{all } \lambda \in C.$$

It is noted that the unit ball of a (p)-space is bounded; therefore an endomorphism of a (p)-space is continuous if and only if it maps the unit ball onto a metrically bounded set. Accordingly, as is well known, the algebra $[E]$ of continuous endomorphisms of E can be made into a complete metric linear algebra with the obvious definition:

6.2. DEFINITION. Let E be a (p)-space, and, for $T \in [E]$, define the norm $\psi(T)$ of T by

$$\psi(T) = \sup \{\psi(Tx) \colon x \in E. \&. \psi(x) = 1\}.$$

6.3. THEOREM. *The algebra $[E]$, with the norm ψ, is a (p)-space.*

Proof. The only point which requires vertification here is that ψ is plurisubharmonic on $[E]$. But, since the mapping $T \to \psi(Tx)$ is plurisubharmonic for any $x \in E$, and $\psi(T)$ is the supremum of a set of such functionals, plurisubharmonicity of ψ is clear.

6.4. LEMMA. *If $T_1 \in E$ and $T_2 \in [E]$, then $\psi(T_1 T_2) \leqq \psi(T_1)\psi(T_2)$.*

Proof. It suffices to verify that $\psi(Tx) \leqq \psi(T)\psi(x)$ for any $T \in [E]$ and any $x \in E$. The inequality is trivially valid if $x = 0$, so let $0 \neq x \in E$. Since $\psi(x) \neq 0$, so that the mapping $\lambda \to \psi(\lambda x)$ is subharmonic and nonconstant on C, it follows that there exists a λ' such that $\psi(\lambda' x) = 1$. Then

$$\psi(T(\lambda' x)) = |\lambda'|^p \psi(Tx) \leqq \psi(T).$$

But $\psi(\lambda' x) = |\lambda'|^p \psi(x) = 1$; therefore $\psi(Tx)/\psi(x) \leqq \psi(T)$, and the lemma follows.

The next result is proved in exactly the same way for $0 < p < 1$ as for $p = 1$; the proof is therefore omitted.

6.5. THEOREM. *Let E be a (p)-space. Then the group $[E]^{-1}$ of invertible elements of $[E]$ is open, and inversion is continuous on $[E]^{-1}$.*

6.6. DEFINITION. The resolvent set $\rho(T)$ of an endomorphism in $[E]$ is defined to be the set

$$\{\lambda \colon \lambda \in C. \&. (\lambda I - T) \in [E]^{-1}\}.$$

If $T \in [E]$ and $\lambda \in \rho(T)$ then $R(\lambda, T) = (\lambda I - T)^{-1}$ is called the resolvent of T. The complement $C\backslash\rho(T) = \sigma(T)$ is called the spectrum of T.

The critical fact to be established next is that the resolvent $R(\lambda, T)$ is a function of class A on $\rho(T)$. Since the Taylor series for $R(\lambda, T)$ is evidently absolutely convergent, this result follows from the lemma below.

6.7. LEMMA. *Let E be a (p)-space. Let the sequence of elements $a_n \in E$ be such that the power series $\sum z^n a_n$ is absolutely convergent for $|z| < r$. Then the power series is ultra convergent on compact subsets of the disk $\{z: |z| < r\}$.*

Proof. The hypothesis is that $\sum\psi(z^n a_n)$ is convergent if $|z| < r$. It is necessary to show that if $r_1 < r$ then for every $\varepsilon < 0$ there exists an n' such that if $n(1), \cdots, n(k) \leqq n'$ and if $t(1), \cdots, t(m) \in [0,1]$ and $\sum t(j) = 1$, and if $|z(1),| \cdots, |z(m)| \leqq r_1$ then

$$\psi\left(\sum_j \sum_i t(j) z(j)^{n(i)} a_{n(i)}\right) < \varepsilon.$$

From the subadditivity of ψ, there results:

$$\psi\left(\sum_j \sum_i t(j) z(j)^{n(i)} a_{n(i)}\right) \leqq \sum_i \psi\left(\sum_j t(j) z(j)^{n(i)} a_{n(i)}\right).$$

Now, $|\sum_j t(j) z(j)^{n(i)}| \leqq r^{n(i)}$; ψ is plurisubharmonic, and therefore subharmonic on the linear span of the singleton $\{a_{n(i)}\}$. From the maximum property of subharmonic functions, there follows:

$$\psi\left(\sum_j t(j) z(j)^{n(i)} a_{n(i)}\right) \leqq r_1^{n(i)p} \psi(a_{n(i)}) \, .$$

Since the series $\sum\psi(r_1^n a_n) = \sum r_1^{np} \, \psi(a_n)$ converges, the lemma follows.

6.7.1. COROLLARY. *Let E be a (p)-space, and let $T \in [E]$. Then the resolvent $R(\lambda, T)$ is a function of class A on the resolvent set $\rho(T)$.*

6.8. THEOREM. *Let E be a (p)-space, and let $T \in [E]$. Then the spectrum $\sigma(T)$ of T is not empty.*

Proof. It is clear that

$$\rho(T) \supset \{\lambda : \lambda \in C. \ \&. \ |\lambda| > (\psi(T))^{1/p}\},$$

and that $R(\lambda, T) \to 0$ as $\lambda \to \infty$. Suppose $\rho(T) = C$, or $\sigma(T)$ is empty. Since ψ is plurisubharmonic, the function $\psi(R(\cdot, T))$ is subharmonic on C, and vanishes at infinity. That function must then vanish everywhere on C. But this is impossible, and the lemma follows.

BIBLIOGRAPHY

1. N. Bourbaki, *Eléments de mathématique*, Livre V, *Espaces vectoriels topologiques*, Chapters I, II, Hermann, Paris, 1953.

2. A. Grothendieck, *Sur certains espaces de fonctions holomorphes*. I, II, J. Reine Angew. Math. **192** (1953), 35–64, 77–95.

3. P. Lelong, *Les fonctions plurisousharmoniques*, Ann. Sci. Ecole Norm. Sup. **62** (1945), 307–338.

4. P. Levy, *Processus stochastiques et mouvement brownien*, Gauther-Villars, Paris, 1948.

5. T. Rado, *Subharmonic functions*, reprint, Chelsea, New York, 1949.

6. A. E. Taylor, *Analysis in complex Banach spaces*, Bull. Amer. Math. Soc. **49** (1943), 652–669.

7. ———, *New proofs of some theorems of Hardy by Banach space methods*, Math. Mag. **23** (1950), 115–124.

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